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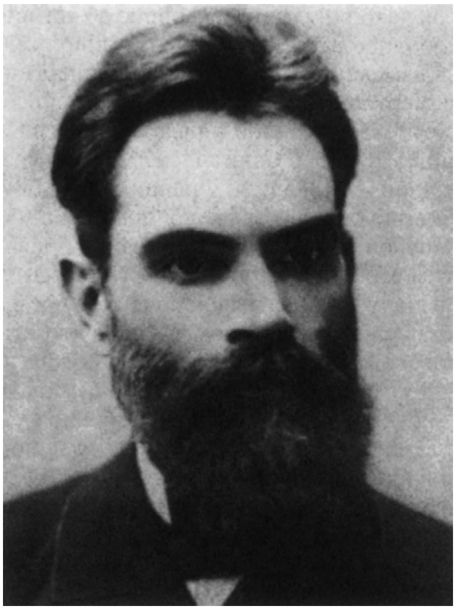
Editorial

ALEKSANDR MIKHAILOVICH LYAPUNOV AND HIS ROLE IN THE ESTABLISHMENT AND DEVELOPMENT OF STABILITY THEORY☆



On the 6 June 2007 we celebrate the 150th anniversary of the birth of Aleksandr Mikhailovich Lyapunov, the eminent Russian mathematician and specialist in mechanics, who established general stability theory and obtained fundamental results in probability theory, the theory of the equilibrium figures of a rotating liquid, potential theory, rigid body dynamics, etc.

1. Brief outline of the life and scientific activity of A. M. Lyapunov

A. M. Lyapunov was born on 25 May (according to the old calendar) 1857 in Yaroslavl into the family of the director of the Demidovsk Lyceum (previously head of the Kazan University observatory). He received his initial education within his family, and then, after the death of his father, Mikhail Vasil'yevich, in 1868, within the family of his uncle, where he was prepared for gymnasium entry together with his first cousin, Nataliya Rafailovna Sechenova, the niece of the eminent physiologist I. M. Sechenov (and whom he was subsequently, in January 1886, to marry). In 1870, together with his mother, Sof'ya Aleksandrovna, and brothers Sergey (1859–1924), a future well-known composer, and Boris (1864–1943), who subsequently became a full member of the USSR Academy of Sciences, specializing in

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'Slav Philology', Lyapunov moved to Nizhnii Novgorod, where he entered the third year of gymnasium, which he left in 1876 with a gold medal. In the same year he entered the Physics and Mathematics Faculty of St Petersburg University, where he was lucky enough to attend lectures given by such eminent scientists as D. I. Mendeleyev and P. L. Chebyshev. With his lectures and then his advice, Chebyshev, as Lyapunov said himself, had a substantial influence on the nature of his subsequent scientific activity.

The first scientific research by Lyapunov was on hydrostatics in his student years under the supervision of Professor of Mechanics D. K. Bobylev. The results he obtained were awarded a gold medal in 1880 by the University Council and formed the basis of his first two publications (in the *Journal of the Physics and Mathematics Society*, St Petersburg, 1881).

After graduating from university (1880), Lyapunov, on the recommendation of Bobylev, remained at the Department of Mechanics to prepare his Master's dissertation. As suggested by Chebyshev, he began to study the problem of determining the equilibrium figures of a rotating liquid, other than ellipsoidal, but encountered difficulties that were, at that time, insurmountable to him. Lyapunov then put aside the solution of Chebyshev's problem and went on to the question of the stability of ellipsoidal equilibrium figures. His comprehensive investigation of this problem using analysis of the modified potential energy of the liquid formed the basis of his Master's dissertation "On the stability of ellipsoidal equilibrium figures of a rotating liquid" which was defended in January 1885 at St Petersburg University, published in 1884 in St Petersburg and immediately became well known in Europe: in 1884 in Germany and 1885 in France, its abstracts were published, and in 1904 it was published in its entirety in France.

In the spring of 1885, Lyapunov was accepted as a privat-docent (unestablished university lecturer) and invited to occupy the chair of mechanics at Khar'kov University. In the autumn of 1885 he moved to Khar'kov, where, to begin with, he had to devote almost all of his time to developing courses, giving lectures and compiling their notes, which were often printed out by lithography. These notes served as the basis of the posthumous publication of his "Lectures on Theoretical Mechanics" (Naukova Dumka, Kiev, 1982).

In 1885–1889, in *Soobshcheniya Khar'kovskogo Matematicheskogo Obshchestva*, Lyapunov published two notes on potential theory and two extensive papers "On the constant screw motions of a rigid body in a liquid" and "On the stability of motion in a special case of the three-body problem". In these two papers there are certain prerequisites of the general theory of the stability of motion, which he began to develop at that time.

In 1892, in Khar'kov, he published his famous monograph "The general problem of the stability of motion", in which the problem of the stability of motion of systems with a finite number of degrees of freedom was examined for the first time in a general formulation with flawless rigour. This monograph was presented by him as a doctoral dissertation, which was defended in September 1892 at Moscow University and, like his Master's dissertation, immediately became well known in Europe: in the same year (1892), in Germany and France, its abstracts were published, and in 1908, in France, it was translated from cover to cover into French.

In 1893, Lyapunov was accepted as a professor in ordinary and, for 10 years (until 1902), continued his teaching and scientific activity at Khar'kov University. During this time he published a range of studies on stability theory, developing and supplementing his previous studies, and several studies on potential theory, rigid body dynamics, etc. In 1900 he began to lecture at Khar'kov University on probability theory, and within 1 year he published two fundamental papers (in *Izvestiya Akademii Nauk* and in *Zapiski Akademii Nauk po Fiziko-matematicheskomu Otdeleniyu*) in which the most general (at that time) formulation of the central limit theorem in probability theory and an entirely new method for proving it were given. This method (the method of characteristic functions) was, in the words of A. N. Shiryayev, "extremely effective in proving the most varied limit theorems, which resulted in its development and wide application".

In 1900 Lyapunov was elected as a corresponding member of the Academy of Sciences, and in November 1901 as a permanent academician in the chair of applied mathematics, which had been vacant since the death of P. L. Chebyshev.

After moving in 1902 to St Petersburg, he was engaged solely in the scientific activity, returned to Chebyshev's problem, considerably broadening its formulation, and over the course of the next 15 years (1902–1917) he obtained an exhaustive solution of this problem. During the same period, he published, in publications of the Academy of Sciences, a range of fundamental studies in an extensive monograph in four volumes on the theory of equilibrium figures of a homogeneous rotating liquid whose particles are mutually attracted according to Newton's law. In these publications, the total volume of which exceeds 1300 pages, he proved for the first time the existence of near-ellipsoidal (but different from ellipsoidal) equilibrium figures, and investigated their stability. Furthermore, he prepared a fundamental work of over 400 pages on the equilibrium figures of an inhomogeneous liquid, which was published (in two parts, in 1925 and in 1927) after his death. As remarked by one of Lyapunov's students, Academician V. A. Steklov, "it was with the same epic feat with which he endeavoured to begin his scientific acivity that he brilliantly ended ... his glorious life".

Lyapunov's scientific contribution received worldwide recognition. In 1908 he took part in the Fourth International Mathematics Congress in Rome; he was one of the editors of the 18th and 19th volumes of the collection of Euler's essays, published in Switzerland; in 1908 he was elected as a foreign member of the Roman Academy of Sciences dei Lincei; in 1916 he was elected as a corresponding member of the Paris Academy of Sciences.

In 1917, Lyapunov, together with his wife, whose health required a change of climate, moved to Odessa. However, the illness persisted, and on 31 October 1918 she died. Lyapunov was unable to bear the death of his adored wife. Having written a note, "We have spent out entire lives together, and we must end them together. I ask that I be buried in the same grave as Nataliya", he shot himself in the head and died 3 days later, on the day of his wife's burial, 3 November 1918.

2. A brief review of Lyapunov's work on stability theory

Lyapunov made a considerable contribution to various areas of mathematics and mechanics (probability theory, the theory of equilibrium figures of a rotating liquid, potential theory, rigid body dynamics, etc.), which brought him truly worldwide fame. A special place among his studies is occupied by his doctoral dissertation "The general problem of the stability of motion", published in 1892 by the Khar'kov Mathematics Society, and published in 1908 in French in the journal *Annales de la Faculté des Sciences de l'Université de Toulouse* (with small changes made by the author).

Without exaggeration it can be said that Lyapunov's doctoral dissertation laid the foundations of a new science – stability theory. He provided a general definition of the stability of motion of mechanical systems with a finite number of degrees of freedom and proposed general methods for investigating the stability of motion of such systems. This definition (Lyapunov stability) and these methods (Lyapunov's first and second methods) relate to any systems that can be described by ordinary differential equations. Moreover, later, Lyapunov's methods were extended to systems that can be described by differential equations with delay or with hysteresis, by partial differential equations, by integrodifferential equations, etc., and were widely used to analyse systems of various kinds in mechanics, physics, chemistry, biology, economics and so on.

Before Lyapunov's studies, rigorous results concerning stability were known only in the case of systems of linear ordinary differential equations (mainly with constant coefficients) and in the case where the stability problem "allows of reduction to a certain problem of maxima and minima" (here and below, unless stated otherwise, Lyapunov's own words are quoted). In the first case, stability was understood as the boundedness of solutions of linear differential equations (Lagrangian stability). The second case relates to the problem of the stability of the equilibrium positions of conservative mechanical systems (Lagrange's theorem, 1788) and, with some stipulations, to the problem of the stability of the steady motions of conservative mechanical systems, allowing of time-independent first integrals (Routh's theorem, 1877). However, the area of application of Lagrange's theorem and Routh's theorem "is extremely limited, and in most cases it is necessary to resort to some other methods".

Before Lyapunov's studies, when investigating the stability of motion, the method of linearization of the equations of perturbed motion in the vicinity of the motion was mainly used, The motion was assumed to be stable if the solutions of the linearized system were described by bounded time functions (Routh, 1877; Thomson and Tait, 1879; Zhukovskii, 1882). In this regard, Lyapunov wrote, "... the method introduces a very considerable simplification", and here "the validity of such a simplification *a priori* has no justification, as the matter reduces to replacing the problem under examination with another to which it may bear no relation ... The only attempt, as far as I'm aware, at a rigorous solution of the problem has been made by Poincaré ... Although Poincaré limited himself to far more general applications .... The ideas contained in the memoir have influenced the greater part of my researches. The task that I set myself, undertaking the present investigation, may be formulated as follows: to indicate those cases in which a first approximation actually solves the stability problem, and to suggest any methods that would make it possible to solve it, at least in some of those cases where, in a first approximation, it is not possible to judge the stability".

To solve the problem, Lyapunov developed two methods, set out in the first chapter of his dissertation.

His first method was based on investigating the solutions of the equations of perturbed motion in the form of a series in positive integer power of arbitrary constants. The first terms of these series correspond to the solutions of the linearized equations of perturbed motion (as pointed out above, before Lyapunov, studies were normally confined to an analysis of these terms only). He also analysed in detail the first terms of the series, and here in the most general case where the linearized equations of motion depend on time in an arbitrary way (generally speaking). This analysis is based on the method of characteristic numbers that he developed. However, not confining himself to an analysis of the first terms of the series only, he investigated in detail the series themselves, studied the conditions of their convergence and as a result demonstrated a general theorem of the stability of arbitrary unsteady motion in a first approximation: "If the system of first-approximation differential equations is correct, and if all its characteristic numbers are positive, then the unperturbed motion is stable". Note that Lyapunov's method of characteristic exponents, which differs from the characteristic exponents introduced by Lyapunov only in sign, is widely used at the present time for the qualitative analysis of dynamical systems.

Lyapunov's second method is based on studying functions (Lyapunov functions) which depend on the phase variables and time and which satisfy, together with its total time derivatives, particular conditions by virtue of the equations of perturbed motion. This method dates back, in its idea, in the words of Lyapunov himself, to the work of Dirichlet (1846), in which a rigorous proof of Lagrange's theorem concerning the stability of the equilibrium position of a conservative mechanical system in the case of a maximum force function was given for the first time, based "on reasoning that can serve as proof of many theorems of this kind".

Lyapunov's theorem of stability, the footnote to it concerning asymptotic stability and two theorems of instability, based on the use of Lyapunov functions, are widely known, have long been classic and form the foundation of Lyapunov's direct method. Note that Lyapunov's theorem on stability and the claim according to the footnote to it concerning asymptotic stability under very general conditions are reversible: if the unperturbed motion is stable or asymptotically stable, then a Lyapunov function can always be found that satisfies his theorem on stability (K. P. Persidskii, 1933) or accordingly the claim concerning asymptotic stability (Massera, 1949). Lyapunov's theorems of instability, based on the use of Lyapunov functions, were later amplified considerably by Chetayev (1934).

The second chapter of Lyapunov's dissertation was devoted to an investigation of the stability of steady motions. Initially he studied in detail a system of linear differential equations with constant coefficients and, with particular assumptions concerning the roots of the characteristic equation of this system, proved the existence of homogeneous forms of phase variables with prescribed properties both of these forms themselves and of their total time derivatives by virtue of a linear system. He then investigated non-linear systems and, on the basis of the second method, using the forms indicated above as Lyapunov functions, proved his famous theorems concerning asymptotic stability or instability for stationary systems in a first approximation. Using his theorem on instability in a first approximation, he rigorously proved for the first time the instability of the equilibrium position of a conservative mechanical system in the case where the force function has no maximum in the equilibrium, and its absence is found from second-order terms in its Taylor series expansion, and for the first time he posed the question of the inversion of Lagrange's theorem: "... will the equilibrium position be unstable if the force function for it is not a maximum?". In the same chapter, Lyapunov investigated in detail the critical cases of a singular zero and a pair of pure imaginary roots.

The third chapter of Lyapunov's dissertation was devoted to an investigation of periodic motions. Initially he examined in detail a system of linear differential equations with periodic coefficients. For such systems, unlike systems with constant coefficients, generally speaking it is impossible to write the characteristic equation in explicit form (for this it is necessary to know the monodromy matrix, i.e. the general solution of the linear system with periodic coefficients). Therefore, above all, Lyapunov developed a method for constructing the coefficients of the characteristic equation in the form of a power series of particular parameters, either present in the system or introduced artificially, and brilliantly realised this method for the case of a second-order linear equation. He then went on to investigate non-linear systems, proved the theorems of asymptotic stability or of instability in a first approximation and studied in detail the critical cases of a single root, equal to unity, and a pair of roots equal in modulus to unity.

In the final section of his dissertation, he examined the case of several zero roots for non-linear quasi-stationary systems of special form and, essentially, proved the theorem of stability in the special case of several zero roots, and also indicated the nature of perturbed motions in this case.

After defending his doctoral dissertation, Lyapunov published papers on stability theory (along with papers on mathematics and other areas of mechanics) for a further 10 years. Six papers (four of them published in *Comptes Rendus des l'Academie des Sciences*, Paris) were devoted to a further detailed analysis of a second-order linear equation with periodic coefficients, the investigation of which was begun in his doctoral thesis. One paper was devoted to an investigation of the critical case of two zero roots, and one was a translation into French of two fragments of Lyapunov's doctoral dissertation. Both of these papers warrant (for various reasons) a more detailed discussion.

In the first of them he investigated the critical case of two zero roots with a Jordan box. As in the analysis of the critical cases considered in his doctoral dissertation, he initially studied the minimum possible order in this case, i.e., a system of two equations. Having made an exhaustive investigation of this case, he published the results in "Mathematical Collection" (1893). From the text of this publication it can certainly be understood that it comprises only the first part of the work he planned to publish. The second part was to contain an investigation of a system of an arbitrary number (more than two) equations in the critical case indicated. However, the second part was not published in his lifetime.

The manuscript of this second part was found in Lyapunov's archive only in 1954, and was not published until 1963 by Leningrad University Press. This work gave an almost exhaustive investigation of stability in the critical case of a pair of zero roots with a Jordan box for a system of an arbitrary number of equations, and, it appears, Lyapunov was not in a position to publish it only because, in one sole subcase, the question remained open. Later it was proved (V. A. Pliss, 1964) that stability occurs in this case.

It is curious that, in proving instability in one of the fully investigated subcases, Lyapunov used a direct method employing a sign-variable function with a sign-definite derivative. Such a function does not satisfy his theorems of instability, and therefore it was necessary to carry out an additional investigation with the aim of giving a rigorous proof of instability. The method of Lyapunov's functions with sign-definite derivatives is now widely used to investigate both asymptotic stability (Ye. A. Barbashin, N. N. Krasovskii, 1952) and instability (N. N. Krasovskii, 1959).

The second of Lyapunov's papers mentioned above was published in French in *Journal de Mathematique Pures et Appliquées* (1897), where he published his theorems on instability both in a first approximation and based on the application of Lyapunov functions. On the basis of these theorems, he proved the instability of the equilibrium position of a conservative mechanical system in cases where, in the equilibrium position, the force function either has no maximum, and its absence is determined from second-order terms, or has a minimum, and this minimum is determined from lower-order terms. In conclusion, he writes that, applying his methods. "it is possible, without doubt, to prove the theorem in question [the inversion of Lagrange's theorem – *AK*] for other cases when there is no maximum. However, can it be proved for the general case?"

This paper provoked a widespread response in Europe: at the turn of the nineteenth and twentieth centuries, a number of papers appeared (Kneser, 1897; Hadamard, 1897; Painlevé, 1897; Hamel, 1903; Silla, 1908; Cotton, 1911) in which special cases of the absence of a maximum force function were discussed in detail for systems with two degrees of freedom, and in 1904 Painlevé gave a mathematical example of a conservative mechanical system whose equilibrium position is stable, in spite of the absence of a maximum force function.

In Russia (the Soviet Union) the idealogical successor of Lyapunov became corresponding member of the USSR Academy of Sciences Nikolai Gur'yevich Chetayev (1902–1959), and, after his death, Chetayev's student, Member of the Russian Academy of Sciences Valentin Vital'yevich Rumyantsev, who observed that "... Lyapunov's researches serve and will serve as an inexhaustable source of creative activity for many generations of applied and pure mathematicians. Their author, without any exaggeration, can be called the pride of Russian science".

In writing this paper, I have used material published in a paper by V. I. Smirnov entitled "The Biography of A. M. Lyapunov" (which appears in the book *A. M. Lyapunov. Collected Papers*, Leningrad: USSR Acad. Sci.; 1948) and in the book by A. S. Shibanov *Aleksandr Mikhailovich Lyapunov* (Moscow: Molodaya Gvardiya; 1985) with a foreword by V. V. Rumyantsev.

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